

■ Energetics with Linear, Isotropic Dielectric Materials

If a situation contains *linear* dielectrics we can use exactly the same linearity argument as in free space electrostatics to get the result that the work necessary to build a "free" charge distribution ρ by dragging charge in from infinity is

$$U = \frac{1}{2} \int_{\text{all space}} \rho \Phi d^3 r.$$

Now, however we have $\vec{\nabla} \cdot \vec{D} = \rho$ and so doing the usual partial integration we get

$$\begin{aligned} U &= \frac{1}{2} \int_{\text{all space}} (\vec{\nabla} \cdot \vec{D}) \Phi d^3 r \\ &= \frac{1}{2} \int_{\text{all space}} (\vec{\nabla} \cdot (\vec{D} \Phi) - \vec{D} \cdot \vec{\nabla} \Phi) d^3 r \\ &= \frac{1}{2} \int_{\text{surface at } \infty} \Phi \vec{D} \cdot d\vec{A} + \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} d^3 r \end{aligned}$$

The potential falls off at infinity at least as fast as $\frac{1}{r}$ and the displacement as $\frac{1}{r^2}$. Since the area grows as r^2 we have for the surface integral the order of $\int_{\text{surface at } \infty} O(1/r) d\Omega \rightarrow 0$ as $r \rightarrow \infty$. We then get

$$U = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} d^3 r.$$

With the assumption of linear, isotropic dielectrics this becomes

$$U = \frac{1}{2} \int_{\text{all space}} \epsilon(\vec{r}) |\vec{E}(\vec{r})|^2 d^3 r$$

Since the permittivity is always positive, we see that $U \geq 0$ with equality if and only if $\vec{E}(\vec{r}) = 0$ for all \vec{r} .

■ Mechanical Forces and Energy

From its definition, the work done by an agent moving something adiabatically (i.e., so slowly that we can ignore kinetic energy) a distance $d\vec{r}$ is $\vec{F} \cdot d\vec{r}$ where \vec{F} is the force applied by the agent. We have used this to calculate the quantity U above and so we can use this expression for energy to work out forces if we move something a little bit. In particular, if we move something in an electrical system, a conductor or piece of dielectric for example, an amount $d\vec{r}$ we can conclude that

$$U^{\text{after move}} - U^{\text{before move}} = \vec{F} \cdot d\vec{r}$$

where \vec{F} is the force the mover must exert. In order to prevent any kinetic energy from being generated, we must have zero *total* force acting on the object so the force on the body due to the electrical system is $-\vec{F}$. For if the stored energy after the move is less than that stored before it, we conclude the mover must restrain the object from moving, i.e., the force the mover must apply is generally in the direction of $-d\vec{r}$, or more precisely, its component in the direction of $d\vec{r}$ is negative. Alternatively, we conclude that the component of force on the body due to the electrical system is positive.

■ Example of two equal point charges separated a distance d in homogeneous dielectric

In this case,

$$\begin{aligned} U &= \frac{\epsilon}{2} \int_{\text{all space}} (\vec{E}_1 + \vec{E}_2)^2 d^3 r \\ &= \frac{\epsilon}{2} \int_{\text{all space}} \vec{E}_1^2 d^3 r \\ &+ \frac{\epsilon}{2} \int_{\text{all space}} \vec{E}_2^2 d^3 r \\ &+ \epsilon \int_{\text{all space}} \vec{E}_1 \cdot \vec{E}_2 d^3 r \end{aligned}$$

The first two terms are independent of d and formally divergent. We keep them carefully segregated so that we can cleanly eliminate them in differences. The cross terms is

easily calculated to be

$$\begin{aligned}
 & \epsilon \left(\frac{q}{4\pi\epsilon} \right)^2 2\pi \int_0^\infty \int_{-1}^1 \frac{(\hat{r} - \frac{d}{2} \hat{e}_z) \cdot (\hat{r} + \frac{d}{2} \hat{e}_z)}{|\hat{r} - \frac{d}{2} \hat{e}_z|^3 |\hat{r} + \frac{d}{2} \hat{e}_z|^3} r^2 dr d(\cos\theta) \\
 &= \frac{q^2}{8\pi\epsilon} \int_0^\infty \int_{-1}^1 \frac{r^2 - (d/2)^2}{((r^2 + (d/2)^2)^2 - (rdx)^2)^{3/2}} r^2 dr dx \\
 &= \frac{q^2}{4\pi\epsilon} \int_0^\infty \frac{r^2 - (d/2)^2}{(r^2 + (d/2)^2)^2 ((r^2 + (d/2)^2)^2 - (rd)^2)^{1/2}} r^2 dr \\
 &= \frac{q^2}{4\pi\epsilon} \int_0^\infty \begin{pmatrix} +1 & , \text{ for } r > d/2 \\ -1 & , \text{ for } r < d/2 \end{pmatrix} \frac{1}{(r^2 + (d/2)^2)^2} r^2 dr \quad (\text{be careful to notice } \sqrt{x^2} = |x|) \\
 &= \frac{q^2}{4\pi\epsilon} \frac{1}{d}
 \end{aligned}$$

Of course this is just the usual potential result. If we move charge 2 by an amount $d\hat{r}$ we get

$$U^{\text{after move}} - U^{\text{before move}} =$$

$$\frac{q^2}{4\pi\epsilon} \left(\frac{1}{|d+dd|} - \frac{1}{d} \right) = \frac{q^2}{4\pi\epsilon} \left(\frac{1}{d} - \frac{dd}{d^2} - \frac{1}{d} \right) = -\frac{q^2}{4\pi\epsilon} \frac{dd}{d^2} = \vec{F} \cdot d\hat{r}$$

If $d\hat{r}$ is orthogonal to the line joining the charges, $dd = 0$ and the force is zero. If it is parallel to the line joining them and in the direction to increase d , then $dd = dr$ is positive and the force provided by the mover is negative. So the electrical force on charge 2 is in the direction to increase d , i.e., repulsive, and has magnitude $\frac{q^2}{4\pi\epsilon} \frac{1}{d^2}$, all of which we well know.

■ Example of moving an uncharged conductor into an existing system of conductors

Now consider the following problem. Suppose we have a collection of n conductors in a homogeneous dielectric ϵ , at potentials Φ_i^b , carrying charges Q_i^b , and no charge densities in the space between the conductors ($\rho = 0$). The superscript "b" means before, and in a moment "a" will mean after. As we know

$$U^b = \frac{1}{2} \sum_{i=1}^n \Phi_i^b Q_i^b.$$

Now let an another conductor, uncharged, be brought from infinity into their midst. Of course

$$U^a = \frac{1}{2} \sum_{i=1}^n \Phi_i^a Q_i^a.$$

There is no direct term for the new conductor because it is uncharged, and so one might be tempted to claim that $U^a = U^b$ but in general this is wrong. In fact the situation is not fully specified for one must say if there any externally induced changes in the charges and/or potentials of the original n conductors.

■ **Holding the charges on existing conductors constant**

To be analogous to the situation with the point charges given above, it is natural to specify that none of the n conductors is allowed to move and that all of them maintain the same charge as they had before, i.e., $Q_i^a = Q_i^b$. In this case, there is no charge moved from infinity to any part of the system and so no work associated with such charge movement. However, under the influence of the new conductor, charges will generally have to move about on the n original conductors and so their potentials will in general change. What is interesting to calculate is $U^a - U^b$. When it is negative, the electrical system will have sucked the new conductor in and if it is positive, the opposite. We will use the electric field formulation of stored energy and Thompson's idea that

$$(\vec{E}^a - \vec{E}^b)^2 = (\vec{E}^b)^2 - (\vec{E}^a)^2 - 2(\vec{E}^a \cdot \vec{E}^b - (\vec{E}^a)^2).$$

Note that the "before" system consists of n conductors and the "after" one, has $(n + 1)$. Choose a volume V that just hugs the surfaces of each of the original n conductors and so does not include the charge on their surfaces but in the "after" system, included all of the new conductor. Of course, the electric field inside it is zero but it has charges on its surface, although their integral vanishes. Now calculate the integral of the second term in the above identity

$$\begin{aligned} & \int_V (\vec{E}^a \cdot \vec{E}^b - (\vec{E}^a)^2) d^3 r \\ &= \int_V \vec{E}^a \cdot (\vec{E}^b - \vec{E}^a) d^3 r \\ &= - \int_V \vec{\nabla} \Phi^a \cdot (\vec{E}^b - \vec{E}^a) d^3 r \\ &= - \int_V \vec{\nabla} \cdot (\Phi^a (\vec{E}^b - \vec{E}^a)) d^3 r + \int_V \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r \\ &= - \int_{\partial V} (\Phi^a (\vec{E}^b - \vec{E}^a)) \cdot d\vec{A} + \int_V \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r \\ &= + \sum_{i=1}^n \Phi_i^a \frac{1}{\epsilon} \int_{\partial V} (\sigma_i^b - \sigma_i^a) dA + \int_V \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r. \end{aligned}$$

The integrals in the sum on the last line simply become $(Q_i^b - Q_i^a) = 0$. The last term can be written as

$$\int_V \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r =$$

$$\int_{V_{\text{except for new cond}}} \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r + \int_{V_{\text{of new cond}}} \Phi^a \vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) d^3 r.$$

the first of these terms vanish since $\vec{\nabla} \cdot (\vec{E}^b - \vec{E}^a) = 0$ in that volume. Similarly the term

$\int_{\text{vol of new cond}} \Phi^a \vec{\nabla} \cdot \vec{E}^b d^3 r$ also vanishes because in this bit of space $\vec{\nabla} \cdot \vec{E}^b = 0$. Finally the last term $\int_{\text{vol of new cond}} \Phi^a \vec{\nabla} \cdot \vec{E}^a d^3 r = \Phi_{n+1}^a \int_{\text{vol of new cond}} \vec{\nabla} \cdot \vec{E}^a d^3 r = \Phi_{n+1}^a Q_{n+1} = 0$ since the new conductor is an equipotential and uncharged.

Consequently we get

$$\begin{aligned} U_{\text{fixed } Q}^{\text{after new cond brought in}} - U_{\text{fixed } Q}^{\text{before}} \\ &= W_{\text{fixed } \Phi}^{\text{mechanical}} \\ &= -\frac{\epsilon}{2} \int (\vec{E}^a)^2 - (\vec{E}^b)^2 d^3 r = -\frac{\epsilon}{2} \int (\vec{E}^a - \vec{E}^b)^2 d^3 r < 0. \end{aligned}$$

Therefore, the mechanical force supplied by the mechanical agent to move the new conductor into the others is generally oppositely directed to the direction of motion. Thus the electrical force on the new conductor always tends to *suck* it into the collection of conductors already there.

■ **Holding the potentials on existing conductors constant**

Another possibility is to hold the potentials on all of the n conductors constant, i.e., $\Phi_i^a = \Phi_i^b$. Then we must take account of the work done by agents that hold these potentials constant because they will in general move some charge from infinity to the conductors. The work done by these electrical agents, as we might call them, is just

$W_{\text{fixed } \Phi}^{\text{electrical}} = \sum_{i=1}^n \Phi_i^b (Q_i^a - Q_i^b)$. We can then say that

$$\begin{aligned} U_{\text{fixed } \Phi}^{\text{after new cond brought in}} - U_{\text{fixed } \Phi}^{\text{before}} &= \frac{1}{2} \sum_{i=1}^n \Phi_i^a Q_i^a - \frac{1}{2} \sum_{i=1}^n \Phi_i^b Q_i^b = \frac{1}{2} \sum_{i=1}^n \Phi_i^b (Q_i^a - Q_i^b) \\ &= W_{\text{fixed } \Phi}^{\text{mechanical}} + W_{\text{fixed } \Phi}^{\text{electrical}} \\ &= W_{\text{fixed } \Phi}^{\text{mechanical}} + \sum_{i=1}^n \Phi_i^b (Q_i^a - Q_i^b). \end{aligned}$$

So we get the interesting result that

$$W_{\text{fixed } \Phi}^{\text{mechanical}} = -\frac{1}{2} \sum_{i=1}^n \Phi_i^b (Q_i^a - Q_i^b) = -\frac{1}{2} W_{\text{fixed } \Phi}^{\text{electrical}}.$$

Now what is the sign of $U_{\text{fixed } \Phi}^{\text{after new cond brought in}} - U_{\text{fixed } \Phi}^{\text{before}}$?

Use the same relation as before but slightly rearranged.

$$(\vec{E}^a - \vec{E}^b)^2 = (\vec{E}^a)^2 - (\vec{E}^b)^2 - 2(\vec{E}^a \cdot \vec{E}^b - (\vec{E}^b)^2)$$

So with the same volume as above we get

$$\begin{aligned} & \int_V (\vec{E}^a \cdot \vec{E}^b - (\vec{E}^b)^2) d^3 r \\ &= \int_V \vec{E}^b \cdot (\vec{E}^a - \vec{E}^b) d^3 r \\ &= - \int_V \vec{E}^b \cdot \vec{\nabla}(\Phi^a - \Phi^b) d^3 r \\ &= - \int_V \vec{\nabla} \cdot (\vec{E}^b (\Phi^a - \Phi^b)) d^3 r + \int_V \vec{\nabla} \cdot \vec{E}^b (\Phi^a - \Phi^b) d^3 r \\ &= - \int_{\partial V} (\Phi^a - \Phi^b) \vec{E}^b \cdot d\vec{A} + \int_V \vec{\nabla} \cdot \vec{E}^b (\Phi^a - \Phi^b) d^3 r \\ &= + \sum_{i=1}^n (\Phi^a - \Phi^b) \int_{\partial V} \vec{E}^b \cdot d\vec{A} + \int_V \vec{\nabla} \cdot \vec{E}^b (\Phi^a - \Phi^b) d^3 r. \end{aligned}$$

The first term in the last line vanishes because $\Phi^a - \Phi^b = 0$ and the second does because $\vec{\nabla} \cdot \vec{E}^b$. Thus we get

$$U_{\text{fixed } \Phi}^{\text{after new cond brought in}} - U_{\text{fixed } \Phi}^{\text{before}} = \frac{\epsilon}{2} \int (\vec{E}^a)^2 - (\vec{E}^b)^2 d^3 r = \frac{\epsilon}{2} \int (\vec{E}^a - \vec{E}^b)^2 d^3 r > 0.$$

$$W_{\text{fixed } \Phi}^{\text{electrical}} > 0, \quad \text{and} \quad W_{\text{fixed } \Phi}^{\text{mechanical}} < 0.$$

We conclude that for either fixed charges or fixed potentials the mechanical forces tend to suck the new conductor in.

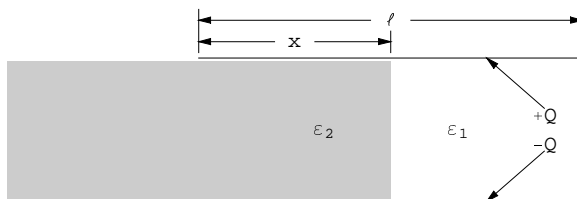
■ **Moving an uncharged piece of a dielectric into an existing system of conductors**

I will not discuss the general case but rather just a simple situation which is easy to do and yields the same general result as the general theorem. Consider a parallel plate capacitor with spacing between the plates small compared to the transverse dimensions of the plates. Imagine a slab of dielectric filling a portion of the gap of height d to a distance x and sticking out beyond the edge of the capacitor. Let the charge on the two plates be $+Q$ and $-Q$ per unit length into the third dimension..

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The electric field, being continuous across the interface between the two dielectrics, is the same everywhere between the two plates. In terms of the as yet unknown potential difference V between the plates, the electric field magnitude is

$$E = \frac{V}{d}$$

and it is directed from the upper plate to the lower one.

Thus in dielectric to the left the displacement field is $\epsilon_2 \frac{V}{d}$ and in the rest, $\epsilon_1 \frac{V}{d}$. Thus the charge density on the top plate is $\epsilon_2 \frac{V}{d}$ over a distance x and $\epsilon_1 \frac{V}{d}$ over the distance $(\ell - x)$. With sign changes this is also the situation on the bottom plate. Thus the total charge per unit length into the third dimension is $\epsilon_2 \frac{V}{d} x + \epsilon_1 \frac{V}{d} (\ell - x) = Q$. Solving for the potential we get

$$V = \frac{Qd}{\epsilon_2 x + \epsilon_1(\ell - x)}. \text{ Finally, the stored charge in the capacitor is}$$

$U_{\text{fixed } Q}(x) = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2 d}{\epsilon_2 x + \epsilon_1(\ell - x)}$. If we increase x by an amount dx , we get for the stored energy change

$$\begin{aligned}
 U_{\text{fixed } Q}^a - U_{\text{fixed } Q}^b &= U_{\text{fixed } Q}(x + dx) - U_{\text{fixed } Q}(x) \\
 &= \frac{dU_{\text{fixed } Q}}{dx} dx \\
 &= -\frac{1}{2} \frac{Q^2 d(\epsilon_2 - \epsilon_1)}{(\epsilon_2 x + \epsilon_1(\ell - x))^2} dx
 \end{aligned}$$

Thus if $\epsilon_2 > \epsilon_1$, the dielectric slab is sucked between the plates by the electrical forces. In the limit as $\epsilon_2 \rightarrow \infty$, we *always* have it sucked in; this is just the conductor case which we did in general above. In the other case, $\epsilon_2 < \epsilon_1$ the slab is expelled from between the plates by the electrical forces.

You may have noticed that electrical equipment, especially that with high voltages gets filthy dirty. This is just the result of dust particles with $\epsilon > \epsilon_{\text{air}}$ getting sucked into an electric field.

What about the case that the plates are connected by a battery or other power supply that keeps V constant? The stored energy then is more naturally expressed as

$$U_{\text{fixed } \Phi}(x) = \frac{1}{2} V Q = \frac{1}{2} \frac{V^2}{d} (\epsilon_2 x + \epsilon_1(\ell - x)).$$

Note that

$$\frac{dU_{\text{fixed } \Phi}(x)}{dx} = \frac{1}{2} \frac{V^2}{d} (\epsilon_2 - \epsilon_1) = \frac{1}{2} \frac{Q^2 d(\epsilon_2 - \epsilon_1)}{(\epsilon_2 x + \epsilon_1(\ell - x))^2}$$

which has the same magnitude but opposite sign from $\frac{dU_{\text{fixed } Q}(x)}{dx}$. The unwary user of this formula might come to the conclusion that the dust particles are repelled rather than attracted, and worry why informal experiments "give the wrong answer". Of course, the answer is that $\frac{dU_{\text{fixed } \Phi}(x)}{dx}$ does not directly give the mechanical force on the dielectric.