Introduction
to
Magnetostatics

■ Qualitative Remarks

People who have taken an E&M course using Purcell's excellent book know that magnetic effects arise from relativistic dynamics. It is the relativistic addition to electrostatic forces when charges move with constant velocity. As we will see, the subject also encompasses also cases in which currents are constant in magnitude but change direction such as going about in a circle. Here we have accelerated charges but experimentally, the fields produced are static - magnetostatic. Understanding how this comes about physically is an interesting problem but all I want to comment on at this time is that often magnetic forces are small corrections to electrostatic ones, of order $(\frac{v}{c})^2$.

If you know the elements of E&M, you can see this by noting that the Coulomb force on a stationary charge q_2 due to another stationary charge q_1 a distance r away is

$\vec{F}_{\text{elec 1 on 2}} = q_2\left(\frac{q_1}{4\pi\varepsilon_0}\,\frac{\vec{e}_{1\to 2}}{r^2}\right)$ (here $\vec{e}_{1\to 2}$ is a unit vector from charge 1 to 2).

If the charges are moving, however, there is another contribution to the force which we call the magnetic interaction (although Purcell shows it arises from the special relativity rule for the transformation of forces) and it is

$$\vec{F}_{\text{mag 1 on 2}} = (q_2\,\vec{v}_2)\times\left(\frac{\mu_0}{4\pi}\;\frac{q_1\,\vec{v}_1\times\vec{e}_{1\to2}}{r^2}\right) = \left(\frac{q_1\,q_2}{4\,\pi\,\varepsilon_0}\;\frac{1}{r^2}\right)\varepsilon_0\,\mu_0(\vec{v}_2\cdot\vec{e}_{1\to2}\,\vec{v}_1 - \vec{v}_1\cdot\vec{v}_2\,\vec{e}_{1\to2}).$$

One very interesting thing about this magnetic force is that it does not satisfy Newton’s third law; if you interchange the indices 1 and 2, you don’t get the same quantity with the sign changed. We’ll come back to this feature at a later time. Today I just want to notice that the ratio of the magnitudes of the magnetic and electric forces is of order $(\frac{v}{c})^2$. This follows from $\varepsilon_0\,\mu_0 = \frac{1}{c^2}$ where c is the velocity of light (I have not shown this but I have used it before and I assume that everyone here knows it). For terrestrial velocities that we are all pretty accustomed to, say 60 miles an hour

$$\frac{v}{c} \simeq \frac{60\text{ (mi/hr)}}{3600\text{ (sec/hr) }186000\text{ (mi/sec)}} \simeq 10^{-7} \qquad \text{and so} \qquad (\frac{v}{c})^2 \simeq 10^{-14}.$$

Pretty small. Even for a velocity that is very large by terrestrial standards, the earths speed about the sun, we have, using the easily remembered numbers that the earth’s orbital radius is 500 light seconds and that a year contains about $\pi\,10^7$ seconds,

$$\frac{v_{\text{Earth}}}{c} \simeq \frac{2\,\pi\,500\text{ light seconds}}{\pi\,10^7} = 10^{-4} \qquad \text{and so} \qquad (\frac{v_{\text{Earth}}}{c})^2 \simeq 10^{-8}.$$

Even the velocity of an electron in a hydrogen atom (being Bohrish about it) is no more than about $10^{-2}\,c$ so magnetic effects are of order 10^{-4} Coulomb ones.

Thus it would seem that magnetic effects are microscopic and can be completely ignored in most practical terrestrial circumstances. It would seem that only when we deal with relativistic velocities need we take magnetic effects into account.

Indeed you may have noticed that this is exactly the situation in gravity. With mass playing the role of charge and a suitable constant G included to conform natural gravitational units to our arbitrary ones (line the ε_0 of electrostatics), Newtonian gravity has exactly the same force law, inverse square, as Coulomb's law, and so we can get a potential satisfying Poisson's equation where there is matter density, and Laplace's equation where there is none. So just as there is a magnetic force in electricity (that we talk a lot about), so also there is a "gravimagnetic" force in gravity (about which you have probably never heard). Surely Einstein's relativistic theory of gravity must contain it and in fact it is the lowest order relativistic correction to Newton's law. *This correction is so small as to be unobservable in any terrestrial experiment - with one exception (sort of).*

The one exception is the precession of the perihelion of Mercury. The amount of precession unaccounted for by Newtonian gravity corrections due to other planets is about 60 seconds of arc per century (I'm being cavalier with the number and chose 60 to make the arithmetic easy). During this time, Mercury makes about 400 orbits of the Sun or moves through an angle os $400 \cdot 360 \cdot 60 \cdot 60$ seconds of arc, and it is out of wack by a mere 60. This is about 1 part in $9 \cdot 10^6$. Compare this with $\frac{c}{v_{\text{Mercury}}} \simeq 25 \cdot 10^6$. So we see that the celebrated precession of Mercury is an effect of order $(\frac{v}{c})^2$.

There is one rather interesting note to make about the gravimagnetic force - its sign is opposite of that the electrical magnetic effect. The reason is that in electricity, like charges repel, whereas in gravity, they attract. This is an intrinsic sign difference and it is not the result os a convention such as the right hand rule. The force term due to magnetism is just the same whether you consistently use a right hand rule, or a left hand rule (E&M is symmetric under parity).

In spite of the similarity in the underlying law, there is no hint in gravitational theory of the rich phenomenology of electricity - equipotential surfaces, conductors, polarized materials, etc. In fact, there is not even the analog of the dipole field (see later).

> The reason we can safely ignore gravimagnetic effects but not magnetic ones is that, for gravity, there is only one sign of charge, whereas in electricity there are both positive and negative charges.

It is this qualitative *physical* difference, plus the fact that matter is composed of electrical structures - atoms - that consist of equal amounts of positive and negative charge, that generates electricity's much richer phenomenology compared to Newtonian gravity. The fact that there is only one sign of gravitational charge means that the lowest order correction to a monopole gravitational field is the quadrupole, whereas in electricity it is dipole. What I mean by this is that you can always make a suitable translation of coordinates in a gravitational problem so as to make the gravitational dipole moment vanish - but this is not always possible in electricity. In fact if the monopole moment vanishes in electricity, there may still be a dipole moment that dominates the field. The gravitational force is always dominated by a $\frac{1}{r^2}$ term, and for a suitably chosen coordinate origin, its lowest order correction vanishes as $\frac{1}{r^4}$. In electrostatics, on the other hand, the lowest order force term may fall as fast as $\frac{1}{r^3}$ (no monopole moment, but a dipole moment.)

Finally, what makes the tiny relativistic correction we call magnetism so important is the same as what makes intrinsic dipoles possible in electricity. A phenomenological electric charge density can vanish exactly in electricity, *but the positive and negative charges may have different velocities.* So we can have situations in which there is exactly zero electrostatic force, but a non-zero magnetic one. In fact the magnetic force can be enormous in such situations, and the little relativistic effect can be powerful indeed.

Important Note
The above remarks are not meant to imply that all of magnetostatics can be derived from relativistic dynamics. Rather they show that magnetism is a reflection of the relativistic symmetry od Maxwell's equations. It is true that in a situation in which one observer sees only an electrostatic field, someone moving w.r.t. the first can see both a magnetic and electric field. This is the circumstance Purcell uses to derive magnetism from electrostatics plus relativistic dynamics. However, it is also perfectly possible for one observer to see only a magnetic field and for a relative mover to see both fields. This possibility is not obvious from the relativistic dynamics argument.

Note that $E^2 - B^2$ is a relativistic invariant so if $B = 0$ in one frame, then E will be bigger than B in any frame. Similarly, if $E = 0$ in one frame, then B will be larger than E in every frame.