The Strüm-Liouville Equation

We know that in curvilinear orthogonal coordinates (q_1, q_2, q_3), Laplace's equation is

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\,\frac{\partial\Phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\,\frac{\partial\Phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\,\frac{\partial\Phi}{\partial q_3}\right)\right)=0$$

where (h_1, h_2, h_3), with $h_i \geq 0$, are the coordinate system's scale factors. In some coordinate systems, this equation has solutions as products of three functions, each depending on just one coordinate and some parameters. Let the generic coordinate be called x, let an index i identify the parameters, and let the functions be $f_i(x)$. The functions that arise in particular coordinate systems have special names – exponentials, Legendre functions, Bessel functions, etc. – but they all satisfy a second order differential equation of the general form

$$\frac{d}{dx}\,(p(x)\,f_i'(x)) + q_{(i)}(x)\,f_i(x) = 0$$

where $p(x) \geq 0$. The parentheses around the index on the function $q_i(x)$ are not part of the name of the function but are included to mean that the usual summing convention on repeated indices does not apply the index in parenthesis. These are called Strüm-Liouville equations. In general for given functions $p(x)$ and $q_i(x)$, this equation will have two linearly independent solutions and I will call them $f_i^1(x)$ and $f_i^2(x)$ (not a square), leaving off the superscript when I wish to speak of either of them. Originating in the Laplacian, pairs of these solutions have properties that are similar to the properties derived using Green's trick with the Laplacian, i.e., you get them by "cross multiplying and subtracting". This gives a perfect differential which is trivially integrated. The first important property is that there is a nice formula relating the two solutions $f_i^1(x)$ and $f_i^2(x)$ so that if you know one of them, you can get the other by a quadrature. Here is how it works.

▪ Finding a second solution $f_i^2(x)$ given a first $f_i^1(x)$

Both functions satisfy the same differential equation:

$$\frac{d}{dx}\,(p\,(x)\,f_i^{1\,\prime}\,(x)) + q_{(i)}\,(x)\,f_i^1\,(x) = 0$$
$$\frac{d}{dx}\,(p\,(x)\,f_i^{2\,\prime}\,(x)) + q_{(i)}\,(x)\,f_i^2\,(x) = 0.$$

"Cross multiply and subtract" to get

$$f_i^2(x)\,\frac{d}{dx}\,(p(x)\,f_i^{1\,\prime}(x)) - f_i^1(x)\,\frac{d}{dx}\,(p(x)\,f_i^{2\,\prime}(x)) = 0.$$

Then notice that

$$\begin{aligned}&\frac{d}{dx}\,(f_i^2(x)\,(p(x)\,f_i^{1\,\prime}(x)) - f_i^1(x)\,(p(x)\,f_i^{2\,\prime}(x)))\\ &= f_i^{2\,\prime}(x)\,(p(x)\,f_i^{1\,\prime}(x)) + f_i^2(x)\,(p(x)\,f_i^{1\,\prime}(x))' - f_i^{1\,\prime}(x)\,(p(x)\,f_i^{2\,\prime}(x)) - f_i^1(x)\,(p(x)\,f_i^{2\,\prime}(x))'\\ &= f_i^2(x)\,\frac{d}{dx}\,(p(x)\,f_i^{1\,\prime}(x)) - f_i^1(x)\,\frac{d}{dx}\,(p(x)\,f_i^{2\,\prime}(x)) = 0.\end{aligned}$$

So there is a constant, call it $-C$, so that $f_i^2(x)\,p(x)\,f_i^{1\,\prime}(x) - f_i^1(x)\,p(x)\,f_i^{2\,\prime}(x) = -C$. Now the only "new" trick is to multiply by $\frac{1}{p(x)\,(f_i^1(x))^2}$ to get $f_i^2(x)\,\frac{f_i^{1\,\prime}(x)}{(f_i^1(x))^2} - \frac{f_i^{2\,\prime}(x)}{f_i^1(x)} = -\left(\frac{f_i^2(x)}{f_i^1(x)}\right)' = -\frac{C}{p(x)\,(f_i^1(x))^2}$ and so we get the nice formula for $f_i^2(x)$:

$$f_i^2(x) = f_i^1(x) \int \frac{C}{p(x)\,(f_i^1(x))^2}\, dx.$$

■ Example - Sinusoids

The simplest example arises in cartesian coordinates where $p(x) = 1$ and we take $q_{(i)}(x) = k^2$. Note that the index i is just identifies k^2 in this case. Then the equation is $f'' + k^2\, f = 0$. One solution is $\sin x$ and the other is proportional to

$$\sin kx \int \frac{C}{\sin^2 kx}\, dx = -\frac{C}{k}\, \sin kx\, \int d\left(\frac{\cos kx}{\sin kx}\right) = -\frac{C}{k}\, \cos kx$$

as we all know.

■ Example - Exponentials

Another example is a real exponential. Take the equation $f'' - k^2\, f = 0$ so that e^{kx} is a solution. A second solution is proportional to

$$e^{kx} \int \frac{1}{e^{2\,kx}}\, dx = -\frac{1}{2\,k}\, e^{kx}\, e^{-2\,kx} \propto e^{-kx}$$

which of course is trivial.

■ Example - Power law

In 2D polar coordinates, we get $\frac{d}{d\rho}\,(\rho\, R') + \frac{k^2}{\rho}\, R = 0$ so that $p(\rho) = \rho$ and $q_{(i)}(x) = \frac{k^2}{\rho}$. One solution is ρ^k and so the other is proportional to

$$\rho^k \int \frac{1}{\rho\,\rho^{2\,k}}\, d\rho = -\frac{1}{2\,k}\, \rho^k\, \rho^{-2\,k} = \begin{pmatrix} -\frac{1}{2\,k}\, \rho^k\, \rho^{-2\,k} & k \neq 0 \\ \log\rho & k = 0 \end{pmatrix} \propto \begin{pmatrix} \rho^{-k} & k \neq 0 \\ \log\rho & k = 0 \end{pmatrix}.$$

■ Example - Sinusoids of a logarithm

Another example is another solution in this case, i.e., $R = \sin(\,k \log\rho)$, which is easily seen to satisfy the differential equation non-trivially when $k \neq 0$. Then the other solution is

$$\sin(\,k\log\rho) \int \frac{1}{\rho\,\sin^2(\,k\log\rho)}\, d\rho = \sin(\,k\log\rho) \int \frac{d(\log\rho)}{\sin^2(\,k\log\rho)} = -\frac{1}{k}\, \sin(\,k\log\rho) \int d\left(\frac{\cos(\,k\log\rho)}{\sin(\,k\log\rho)}\right) \propto \cos(\,k\log\rho).$$

In spherical coordinates this equation relates various Legendre functions, and in cylindrical coordinates, various Bessel functions. Similar relations arise for the functions suitable for other coordinate systems.

■ Oscillatory behavior of some Strüm-Liouville solutions

A very important property of the Strüm-Liouville solutions is that, in some cases, they are oscillatory. Consider the special case that $q_i(x) > 0$. Then it is easy to see that $f_i(x)$ is oscillatory (except possibly if both $p(x)$ and $f_i(x)$ asymptotically approach zero, which case I will ignore). When the function is positive, then $(p(x)\, f_i'(x))$ is decreasing. This could be due to $p(x)$ decreasing and $f_i'(x)$ increasing more slowly; however, $p(x)$ is positive (and I have assumed it does not approach zero asymptotically), so it cannot **always** be decreasing. Eventually we will get to a region where $p(x)$ is no longer decreasing and there $f_i'(x)$ must be decreasing. Eventually then the sign of $f_i(x)$ will change and so we can conclude from the above argument again that $f_i'(x)$ will begin increasing. And so $f_i(x)$ will be oscillatory and develop an infinite number of zeros.

■ Orthogonality

The most important property of the solutions of the Strüm-Liouville equation is their orthogonality if certain conditions are satisfied. Consider the solutions $f_i(x)$ and $f_j(x)$ where the two indices will be different. Then we have

$$\frac{d}{dx}\,(p(x)\, f_i'(x)) + q_{(i)}(x)\, f_i(x) = 0$$
$$\frac{d}{dx}\,(p(x)\, f_j'(x)) + q_{(j)}(x)\, f_j(x) = 0.$$

With the usual trick we immediately get

$$\frac{d}{dx}\,(f_j(x)\, p(x)\, f_i'(x) - f_i(x)\, p(x)\, f_j'(x)) = (q_{(j)}(x) - q_{(i)}(x))\, f_i(x)\, f_j(x).$$

Integrate this from $x = a$ to $x = b$ to get

$$p(b)\,(f_j(b)\, f_i'(b) - f_i(b)\, f_j'(b)) - p(a)\,(f_j(a)\, f_i'(a) - f_i(a)\, f_j'(a)) = \int_a^b (q_{(j)}(x) - q_{(i)}(x))\, f_i(x)\, f_j(x)\, dx.$$

The interesting situation is when the limits a and b are chosen such that for any i and j,
with $i \neq j$, the left-hand side vanishes. For convenience let $q_{(j)}(x) - q_{(i)}(x) = w_{ij}(x)$. Then we have an orthogonality relation, namely, for any $i \neq j$, satisfying the above condition, $\int_a^b w_{ij}(x)\, f_i(x)\, f_j(x)\, dx$. The function $w_{ij}(x)$ is called a *weight* function and it often has only a trivial dependence on the indices i, j. The usual case that $w_{ij}(x) = C_{ij}\, w(x)$ where C_{ij} is a constant. In this case we get

$$\int_a^b w(x)\, f_i(x)\, f_j(x)\, dx = 0 \text{ for special values of } a \text{ and } b.$$

Usually, to make the solutions definite, some standard normalization is adopted, often (but not always) so that $\int_a^b w(x)\,(f_i(x))^2\, dx$ is convenient.

■ Example - Sinusoids

Again, the simplest example is when $p = 1$ and $q_n = n^2$ where n is an integer. Then $w_{nm}(x) = n^2 - m^2$ and we can drop it from the orthogonality relation; the weight function is taken as 1. We have the sinusoid equation, $f_n''(x) + n^2\, f_n = 0$, whose solutions are $\sin nx$ and $\cos nx$. Thus if we choose $a = 0$ and $b = 2\pi$ (the period of the oscillatory sinusoids) we get $\int_0^{2\pi} \sin nx \sin mx\, dx = 0$ if $n \neq m$, and similarly for other combinations.

■ Singular points of the Strüm-Liouville equation

There is another situation that arises in important examples, *e.g.*, spherical polar coordinates. The zero in the orthogonality equation arises because the p function vanishes at one or both ends, i.e., ($p(a) = 0$, $p(b) \neq 0$) or ($p(a) \neq 0$, $p(b) = 0$) or $p(a) = p(b) = 0$. Points at which p vanishes are called the "singular points" of the Strüm-Liouville equation and although they are of great importance in the theory, I will make only a few remarks about them. At singular points, you can see that the coefficient of the second derivative in the differential equation vanishes and so the order of the equation changes at the point. It turns out that, except in special circumstances, solutions are singular at singular points. More generally, if you assume a Taylor's series solution of the differential equation, the series will converge in a circle in the complex plane centered on the expansion point and extending out to the nearest singular point. However, you can assume a Taylor's series expansion about a singular point itself, and so force regularity at it; such solutions are often of great importance. Examples are the ordinary Bessel functions $J_m(x)$ given as a series in $x - x = 0$ is a singular point for the Bessel equation. Notice that their companion functions $N_m(x)$ however are singular at the singular point.

But any further discussion of these matters should be left to a mathematical physics course.