

1 Zero-point energy

The electromagnetic field may be thought of as the integral over a spectrum of harmonic oscillators, corresponding to the wavelengths of all possible modes (= photons). Why can we say this? Let's write down the Schrödinger equation in free space:

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi \quad (1)$$

This is, as we know, just the classical Newtonian equation of motion

$$p^2/2m = E \quad (2)$$

with the replacement of \vec{p} and E with the operators

$$\vec{p} = -i\hbar\vec{\nabla} \quad E = i\hbar\frac{\partial}{\partial t} \quad (3)$$

Changing the Newtonian equation of motion to the (special) relativistic one:

$$E^2 = \vec{p}^2 + m^2 \quad (4)$$

one obtains

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi \quad (5)$$

in natural units, the well known Klein-Gordon equation. Writing the K-G equation as

$$\left[\frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2) \right] \phi(\vec{p}, t) = 0 \quad (6)$$

one can see that the Klein-Gordon equation has the form of an equation of motion for a simple harmonic oscillator with frequency:

$$\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} \quad (7)$$

Refreshing your memory from introductory QM, the Hamiltonian for the harmonic oscillator can be written as

$$H = \omega(a^\dagger a + \frac{1}{2}) \quad (8)$$

where

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad p^2 = -\frac{\omega}{2}(a - a^\dagger)^2 \quad (9)$$

and the energy levels are

$$E_n = \omega(n + \frac{1}{2}) \tag{10}$$

where n is a whole number. That $\frac{1}{2}$ is the heart of the whole story.

Taking all possible mode frequencies in the Klein-Gordon equation, the total energy is

$$E = \int d^3\vec{p} \omega_{\vec{p}}(a^\dagger a + \frac{1}{2}) \tag{11}$$

Define the vacuum as the state with no photons in any mode. The vacuum energy is

$$2 \times \frac{1}{2} \int d^3\vec{p} \omega_{\vec{p}} \tag{12}$$

(The factor of 2 is from the 2 photon polarizations.)

Bit of a problem here — this is infinite. Infinities are obviously unphysical. Such an infinite energy shift could not be detected by typical means, since experiments measure only *differences* from the ground state. However, in general relativity, gravity couples directly to energy. An overall vacuum energy such as this would contribute directly to the cosmological constant. But the cosmological constant is manifestly not infinite.

One major quibble with our derivation above would be the fact that, when we claimed the integral was infinite, we were taking the upper limit of the momentum integral to be infinity. That is perhaps not fair. We know that we don't know what happens above the Planck scale $E_{pl} = 1/\sqrt{G}$ (it is a "known unknown") so perhaps we should just cut off the integral at that point (a sort of Debye cutoff, indicating that the continuum of the vacuum may break down at that scale, somewhat like the continuum approximation to a solid breaks down at the atomic spacing scale). What would we expect the cosmological constant to be if this is the case?

The Planck scale is at $\mathcal{O}(10^{19} \text{ GeV})$. If we use that as the cutoff for the integral in eqn. (12), we get that the energy density (which has units of energy⁴) $\propto E_{pl}^4 = \mathcal{O}(10^{76} \text{ GeV}^4)$. This is an enormous number, just as clearly unphysical as the infinity was.

Maybe going up to the Planck scale was a bit ambitious. Maybe there is just some physics below the Planck scale that we don't understand. But we *do* think we understand things at least up to $\sim 100 \text{ GeV}$, since we do experiments (at accelerators) up to that energy all the time, and we don't see anything too different from our expectations at those energies. But if we were to take the cutoff at just above 100 GeV, the vacuum energy density would be $\mathcal{O}(10^8 \text{ GeV}^4)$, also in disagreement with reality. The observed energy density in the vacuum ("dark energy"), taken from information combined

from type Ia supernovae, the CMB, cluster density, etc., is $\mathcal{O}(10^{-48} \text{GeV}^4)$. A long way off.

Could there be something that would cause the vacuum energy density to cancel – something that could give an opposite sign? This is an intriguing possibility. Fermionic fields actually do contribute an opposite sign to the bosonic field vacuum energy above. So perhaps supersymmetry is the answer. As noted by Zumino in 1975 (at the dawn of supersymmetry), a globally supersymmetric theory would indeed have vanishing cosmological constant. The problem is, in nature, supersymmetry is clearly not an exact symmetry. We do not detect supersymmetric particles below 100 GeV, thus the symmetry breaking must be at least at that scale. Thus we are left with the same problem as the previous paragraph.

2 The Casimir effect

Maybe our notion of zero-point energies is mistaken. Maybe there's no reason that they should contribute to vacuum energy because maybe they just don't exist (perhaps our understanding of how fields are quantized is in error with regard to those zero-point energies). Do we have any direct proof that they exist?

We do have such direct evidence, in the form of the Casimir effect.

Consider two parallel conducting plates spaced a distance d apart.

Go to Casimir Effect paper...