1 Antiparticles

The Klein-Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi \tag{1}$$

that we derived in the previous lecture is not satisfactory for dealing with massive particles that have spin. Such an equation must take into account both spin states. The wavefunction must be a spinor rather than a scalar, and the spin states are *coupled* — thus a simple scalar equation like the Klein-Gordon equation just won't do.

So, in the 1920s, Dirac set out to find a wave equation that would describe the electron.[1] He mathematically showed that in order to avoid negative probability density solutions, his equation would have to be *linear* in $\frac{\partial}{\partial t}$, unlike the K-G equation, which has a $\frac{\partial^2}{\partial^2 t}$. Thus his equation would have to be of the form

$$H\phi = (\vec{\alpha}\cdot\vec{p} + \beta m)\phi \tag{2}$$

but must also satisfy the relativistic energy-momentum relation

$$H^2\phi = (\vec{p}^2 + m^2)\phi \tag{3}$$

The question is, what are $\vec{\alpha}$ and β? Looking at this, you can see that they can't just be real numbers. It turns out that the simplest representation of $\vec{\alpha}$ and β is a set of 4x4 matrices. If one takes the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and then defines the set of 4x4 matrices:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5}$$

where I is the 2x2 identity matrix, one satisfies both equations (2) and (3). Thus we have the Dirac equation — it is just equation (2) with $\vec{\alpha}$ and β described by (5). But how can we interpret the 4 elements of the resulting

[1] Dirac's motivation was actually not to describe spin, but to avoid negative probability density solutions of the Klein-Gordon equation. But his work ended up not only describing spin, but finding an explanation of the Klein-Gordon negative probability density solutions — as we will see, antiparticles.

wavefunction eigen-spinor? If it were 2 elements, that would be simple – it would just be the +1/2 and -1/2 spin components. But 4?

Let's consider a free space eigen-spinor solution with momentum $\vec{p}$, which will have the form $\phi = u(\vec{p})e^{-i\vec{p}\cdot\vec{x}}$. The Dirac equation becomes

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} u_{1,2} \\ u_{3,4} \end{pmatrix} = E \begin{pmatrix} u_{1,2} \\ u_{3,4} \end{pmatrix} \tag{6}$$

There are 4 solutions to this. If we take

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

then the solutions are

$$\begin{aligned} u_1 = N \begin{pmatrix} \chi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1 \end{pmatrix}, &\qquad u_2 = N \begin{pmatrix} \chi_2 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_2 \end{pmatrix}, \\ u_3 = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi_1 \\ \chi_1 \end{pmatrix}, &\qquad u_4 = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\chi_2 \\ \chi_2 \end{pmatrix}. \end{aligned} \tag{8}$$

The first 2 have associated energy eigenvalue $E = \sqrt{\vec{p}^2 + m^2}$. But the second 2 have energy eigenvalue $E = -\sqrt{\vec{p}^2 + m^2}$ — a negative energy! What is the interpretation of these solutions?

Consider an electron of energy E and 3-momentum $\vec{p}$. Its electromagnetic 4-vector current is

$$j^\mu(e^-) = -2e|N^2| \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}. \tag{9}$$

Now consider an antiparticle, a positron, with the same E, $\vec{p}$. Since its charge is $+e$,

$$j^\mu(e^+) = +2e|N^2| \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = -2e|N^2| \begin{pmatrix} -E \\ -p_x \\ -p_y \\ -p_z \end{pmatrix}, \tag{10}$$

which is the same as the current j^μ for an electron with $-E$, $-\vec{p}$. Thus, an emission of a positron with energy E is the same as the absorption of an electron of energy $-E$.

Thus the negative-energy states in the Dirac equation can be interpreted as positive-energy *antiparticles*.

Or vice-versa. One especially interesting thing about the Dirac equation is that it puts particles and antiparticles on the same footing. There is a complete symmetry between them as far as the Dirac equation is concerned. The fact that we had particles be the first 2 solutions and antiparticles the last two was just an example of our personal prejudice — we could just as easily have defined $\vec{\alpha}$ and β so that antiparticles were the first 2 solutions and particles were the last 2.

But when we look around us, we see only particles. Antiparticles were discovered a few years after Dirac published his equation, and we make them every day in laboratories, etc., but they are rare in the natural world. But mathematically there is a symmetry between particles and antiparticles. So, **why do we see only particles?**

2 Baryonic freezeout

The hot big bang model of the universe implies that at an early epoch of the universe, leptonic and baryonic pairs existed in a fully mixed state in equilibrium with radiation. As the universe expanded and cooled, matter and antimatter was continually being created and annihilated,[2] but eventually freezes out into a steady-state number density. This phase transition occurs when the expansion rate of the universe exceeds the annihilation rate, the latter of which is determined by the matter-antimatter annihilation cross-section. We can use the same method we used two weeks ago for dark matter neutrinos and neutralinos to determine the freeze-out density of protons and antiprotons.

Unlike neutrinos and neutralinos, protons and antiprotons annihilate via the strong interaction, via diagrams such as

[2]There will be time to murder and create
And time for all the works and days of hands
That lift and drop a question onto your plate.
— T.S. Eliot (1917)

The carrier of the strong force is the gluon, but at low energies such as this, the coupling constant α_s is large, and instead of large numbers of gluons, the intermediary particle can be thought of as the π^0.[3] Following the analysis of p. 9-11 of the second week's lecture notes, the cross-section is $\sigma \sim \alpha_s^2/m_{\pi^0}^2$. Thus, analogous to the case of neutralinos, we have

$$n_p \propto e^{-m_p/T} \int_0^\infty p^2 dp e^{-p^2/(2m_pT)} \propto (m_pT)^{3/2} e^{-m_p/T} \tag{11}$$

and we must solve the equation

$$\frac{\alpha_s^2}{m_{\pi^0}^2} m_p^3 x^{-3/2} e^{-x} = \frac{m_p^2}{m_{\rm Pl} x} \tag{12}$$

and we obtain

$$x \approx \ln\left(\frac{\alpha_s^2 m_p m_{\rm Pl}}{m_{\pi^0}^2}\right) - \frac{1}{2}\ln\ln\left(\frac{\alpha_s^2 m_p m_{\rm Pl}}{m_{\pi^0}^2}\right) \approx 50 \tag{13}$$

where $m_p \approx 1$ GeV, $m_{\pi^0} \approx 100$ MeV, and $\alpha_s \approx 1$. Thus, protons and antiprotons freeze out at a temperature $T \sim m_p/50 \approx 20$ MeV.

Again analogous to the case of neutralinos, we expect a post-freezeout abundance

$$\frac{n_p}{n_\gamma} \approx \frac{H_f/\sigma v}{n_\gamma} = \frac{T_f^2/m_{\rm Pl}}{(\alpha_s^2 m_p^2/m_{\pi^0}^4) m_{\rm Pl} T_f^3} \approx 5 \times 10^{-22} \tag{14}$$

where H_f and T_f are the values of the Hubble constant and temperature at freezout respectively. Each proton weighs approximately 1 GeV, so this corresponds to a mass density (using $n_\gamma = 400$ cm^{-3} today) of $\Omega_{\rm baryon} \sim 4 \times 10^{-13}$ today.

But this is completely, utterly different than what we observe! We see that baryons make up approximately 4% of the closure density of the universe, not 4×10^{-13} of it. We know this with excellent precision from both BBN and the CMB. So why do we have so many more baryons than we expect? And where did all the antibaryons go?

3 Matter and antimatter domains?

Perhaps we are just in a region of the universe that is metter-dominated, and other regions are dominated by antimatter. This picture can be compared

[3] The π^0's quark content is $\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})$, so it acts sort of like a creation and annhiliation operator for light quarks.

to the spontaneous formation of ferromagnetic domains when a piece of unmagnetized iron cools below the critical temperature in the absence of a magnetic field.[1] As to the mechanism for how such a scenario could be realized, we will not need to delve into here, because we can show that it is excluded by a combination of information from the CMB and from the CDG (cosmic diffuse gamma radiation) spectrum.

We know that if matter and antimatter domains exist, they must be in contact. This is because if there were voids in between the domains, such voids would show up as anisotropies in the CMB.

Since the domains are in contact, there should be annihilation between the nucleons and antinucleons at the domain boundary. The main reactions are proton-antiproton and neutron-antineutron annihilations to several pions and photons, *ie.* $p+\bar{p}$ and $n+\bar{n}$ to multiple π^+, π^-, π^0, and γ. The $\pi^\pm$ decay primarily to $\mu+\bar{\nu}_\mu$, and the μ then decay to $e\bar{\nu}_e\nu_\mu$. The π^0 decay to $\gamma+\gamma$. There will thus be two potentially observable effects from these interactions. The electrons from the charged pion decays will Compton scatter off CMB photons, which could contribute to the so-called Sunyaev-Zeldovich effect, and will also heat the interstellar medium. Both of these effects could alter the CMB spectrum. The photons from the π^0 decays will contribute to the CDG spectrum. An analysis of the size of these effects shows that the effect on the CMB from the electrons is negligible compared with experimental sensitivity. However, as shown by Cohen, de Rujula, and Glashow in 1998, the effect from the photons on the CDG spectrum places quite significant limits on the size of matter-antimatter domains [2]. The following figure shows the observed CDG spectrum, with curves showing the expected contribution from matter-antimatter annihilation photons produced by domains of size 20 Mpc (upper curve) and 1 Gpc (lower curve). The data thus rules out domain sizes smaller than a Gpc, *ie.* approximately the size of the universe. This effectively rules out matter-antimatter domains.

There are still quibbles with this argument, since effects such as magnetic fields at the domain boundaries can effect the amount of matter-antimatter annihilation. Experiments are still being done to both obtain improved sensitivity to the CDG and to look for antinuclei, such as antihelium, that could drift across domain boundaries. However, the majority of people in the field feel that models that would avoid the CDG limits tend to be rather contrived, and people would be extremely surprised if antinuclei experiments such as BESS and AMS were to see a signal. (Needless to say, so far they have not.)

So we are left with the fact that something in the early universe must have produced an overall asymmetry between baryons and antibaryons.

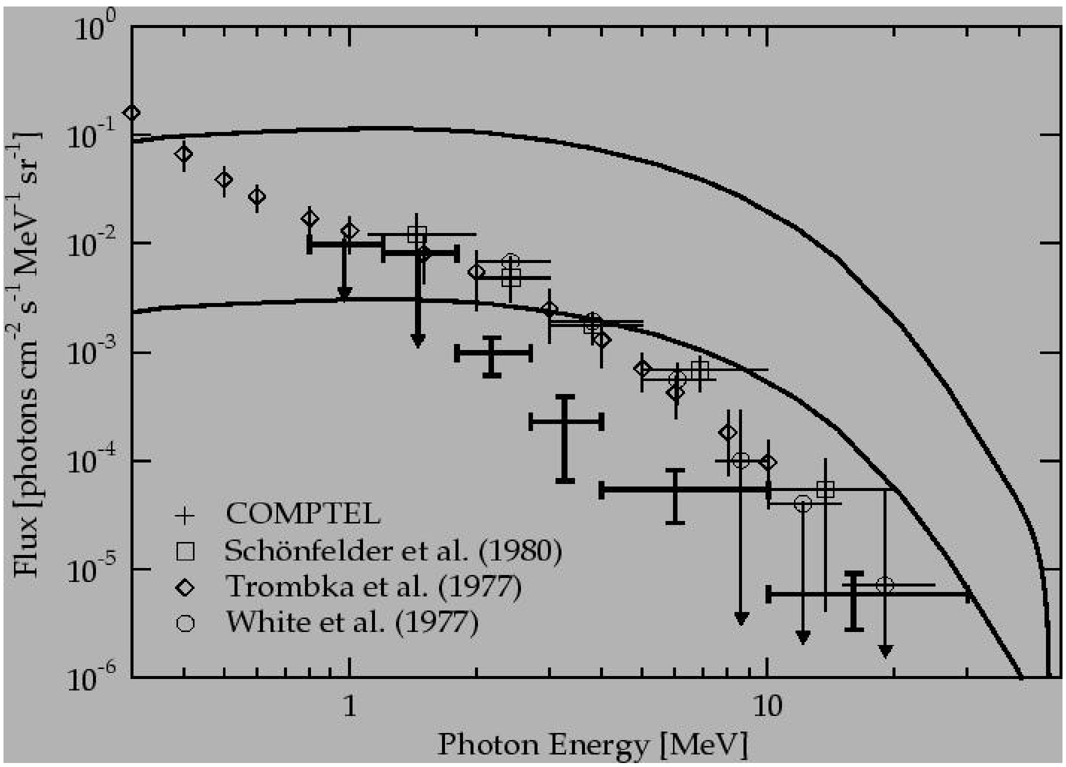

Figure 1: The observed cosmic diffuse gamma radiation (CDG) spectrum, with curves showing the expected contribution from matter-antimatter annihilation photons produced by domains of size 20 Mpc (upper curve) and 1 Gpc (lower curve). The data thus rules out domain sizes smaller than a Gpc [2].

What fundamental physics could cause such an asymmetry? This questions is known as the baryogenesis problem. In order to explain what sort of physics could solve it, we will need to learn about the discrete symmetries of the Standard Model.

4 Discrete symmetries

The Standard Model Lagrangian contains many symmetries. It of course obeys special relativity, thus it is symmetric under Lorentz transformations. It also conatins the so-called "gauge" symmetries of the interactions. For example, electromagnetics is symmetric under a redefinition of the scalar and vector potentials

$$\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned} \tag{15}$$

where λ is *any* scalar function. In group theory terms, this symmetry is called a $U(1)$ symmetry. The weak and strong interaction contribute $SU(2)$ and $SU(3)$ gauge symmetries respectively, thus the Standard Model gauge symmetry is $SU(3) \times SU(2) \times U(1)$.

In addition to these continuous symmetries, there are three independent discrete transformations that also preserve the Minkowski interval $t^2 - \vec{x}^2$. They are the charge conjugation operator (C), the parity operator (P), and the time-reversal operator (T). These form a complete set of discrete Minkowski interval-preserving transformations of the Hilbert space. Although other discrete interval-preserving transformations exist in the Standard Model, all can be formed from C, P, T, and the group of continuous Lorentz and gauge rotations.

Maxwell's equations, and thus the electromagnitic interaction, are symmetric under each of the three discrete transformations. Detailed studies of the magnetic dipole moment of the neutron show that the strong interaction is symmetric under each of the 3 transformations as well. General relativity is also symmetric under each of the 3 interactions. But what about the weak interaction?

5 The Sakharov conditions

In 1967, Sakharov showed that 3 conditions are required for

6 The weak interaction and CP violation

7 Baryon number violation

8 Supersymmetry and CP violation

9 Leptogenesis

References

[1] F.W. Stecker, hep-ph/0207323.

[2] A.G. Cohen, A. deRujula, and S. Glashow, *Astrophys. J.* **495**, 539 (1998).