Oscillatory motion

Equations of motion

\[ \Rightarrow \text{constant force (not oscillatory)} \]

\[ \Rightarrow F \propto -x \]

\[ \Rightarrow F_{\text{drag}} \]

Constant force

[A motivational example]

\[ ma = F \quad \text{with} \quad F = \text{const., perhaps} -mg \]

\[ m \ddot{x} = -mg \]

\[ \ddot{x} = -g \]

\[ \Rightarrow x(t) = -\frac{1}{2} gt^2 + v_0 t + x_0 \]

"Equations of motion"

- Contains all possible \( x(t) \) for the system
- Two constants \((x_0, v_0)\) determined by initial conditions
- Solns works for any constant force (perhaps electrostatic, etc.)

\[ \rightarrow \text{All parabolas in (x,t) space} \]
Simple harmonic motion \((F \propto -x)\)

Same approach:

\[
ma = F \quad \text{with} \quad F = -kx \quad \text{(spring, for example)}
\]

\[
\ddot{x} = -\frac{k}{m} x
\]

Equ. of motion \(\Rightarrow\) done!

But would like to know what \(x(t)\) looks like.

Can't just integrate this one: "differential equation"

In general: tough to solve analytically (often impossible)

Valid approach: guess \(x(t)\) and see if it works.

\(\Rightarrow\) "Ansatz"

Need something whose 2\text{nd} deriv. is proportional to itself.

Try: \(x = A \cos (\omega_0 t)\)

\[
\dot{x} = -A \omega_0 \sin (\omega_0 t)
\]

\[
\ddot{x} = -A \omega_0^2 \cos (\omega_0 t)
\]

\[
\Rightarrow \ddot{x} = -\omega_0^2 x \quad \Rightarrow \text{need } \omega_0 = \sqrt{\frac{k}{m}} \text{ for this to work}
\]

Also works:

\[
x = A \sin \omega_0 t
\]

\[
x = A \cos \omega_0 t + B \sin \omega_0 t
\]

\(\Rightarrow\) verify (next page)
Take $x = Ax_1 + Bx_2$ with $x_1$ & $x_2$ both satisfying our eqn. of motion, e.g.:

$$x_1 = -\omega_0^2 x_1$$

$$\Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

Then:

$$\ddot{x} = A\ddot{x}_1 + B\ddot{x}_2$$

$$= A(-\omega_0^2 x_1) + B(-\omega_0^2 x_2)$$

$$= -\omega_0^2 (Ax_1 + Bx_2)$$

$$\Rightarrow x$$

$$\Rightarrow x = -\omega_0^2 x \Rightarrow \text{works!}$$

So,

$$x(t) = A\cos \omega_0 t + B\sin \omega_0 t$$

from initial conditions

A general solution because:

1. It works, and
2. It has two "constants of integration"

With some trig identities:

$$x(t) = C \cos (\omega_0 t + \delta)$$

Also a general solution for some reasons

all okay $x(t)$

$\Rightarrow$ just different initial conditions

Demos: spring on force probe

"different motions"

"with $m \to 4m"
Pendulum

From last time:

\[ T = I \ddot{\theta} \]
- \( mgR \sin \theta = I \dot{\theta} \)

\[ \Rightarrow \ddot{\theta} = -\frac{mgR \sin \theta}{I} \]

Torque acting in opposite direction to \( +\theta \)

For simple pendulum,

\[ \ddot{\theta} = -\frac{g}{L} \sin \theta \]

\[ \ddot{\theta} = -\frac{g}{2} \sin \theta \]

Not simple harmonic, but if \( \theta \ll 1 \),

\[ \sin \theta \approx \theta \]

\[ \Rightarrow \ddot{\theta} \approx -\frac{g}{2} \theta \]

or

\[ \ddot{\theta} = -\omega_0^2 \theta \]

\[ \omega_0 = \sqrt{\frac{g}{L}} \]

Done!

All the work already done for spring case.

Demos:

\[
\begin{cases}
\omega_0 = \omega_0' & \text{if } \theta \text{ small} \\
\omega_0 \neq \omega_0' & \text{if } \theta \text{ large} \\
\text{walking} &
\end{cases}
\]

Damping

\[ F = -kx - \gamma \dot{x} \]

"spring" drag, prop. to \( \dot{x} \)

2nd law:

\[ m\ddot{x} = -kx - \gamma \dot{x} \]
or

\[ \ddot{x} + \beta \dot{x} + \omega_0^2 x = 0 \]

\[ T_2 \dot{x} + T_3 \frac{k}{m} \]

Can we find \( x(t) \)?

To help our guess:

- \( \beta = 0 \Rightarrow \sinh \) (cosine)
- \( \omega_0 = 0 \Rightarrow \) exponential

\[
\begin{align*}
  \dot{v} &= -\beta v \\
  v &= \exp(-\beta t) \\
  \Rightarrow \quad x &= x_0 \exp(-\beta t)
\end{align*}
\]

So, we want a mix of \( \exp(\cdot) \) & \( \cos(\cdot) \).

Not sure what decay const or ang. freq. to guess, so leave general for now and see what works.

Try: \( x(t) = C e^{-\beta t} \cos(\omega_1 t + \delta) \)

1. Find \( x, \dot{x} \)
2. Plug into \( \ddot{x} + \beta \dot{x} + \omega_0^2 x = 0 \)
3. Will get a bunch of terms with \( \sin(\cdot) \) & \( \cos(\cdot) \) bunches with \( \cos(\cdot) \)
4. Group:

\[
(2bu_1 - u_1 \beta) C e^{-\beta t} \sin(\omega_1 t + \delta)
+ (\omega_0^2 - \omega_1^2 + b^2 - b\beta) C e^{-\beta t} \cos(\omega_1 t + \delta) = 0
\]

5. This, need

\[
\begin{align*}
  2bu_1 - u_1 \beta &= 0 \\
  \omega_0^2 - \omega_1^2 + b^2 - b\beta &= 0
\end{align*}
\]

\[
\begin{align*}
  \text{Need:} & \quad b = \frac{\beta}{2} \\
  & \quad \omega_1^2 = \omega_0^2 - \frac{b^2}{4}
\end{align*}
\]
So, general solution:

\[
x(t) = C \cdot e^{-\frac{\beta t}{2}} \cos(\omega_1 t + \delta)
\]

\[
\Rightarrow \omega_1 = \sqrt{\omega_o^2 - \beta^2}\]

\[\beta < 2\omega_o \text{ case:} \]

\[\beta = 2\omega_o: \]

\[\beta > 2\omega_o: \]

"under-damped"

"critically damped" 
\[(\omega_1 = 0)\]

"over-damped"
\[(\omega_1 \text{ imaginary})\]

Demos:
- tuning fork on scope
- air track with varying \( v \)