Position, velocity, acceleration
Freefall
2D motion
Projectiles

Position, velocity:

Velocity = rate of change in position

Over \([t_1, t_2]\), define "average velocity" as:

\[ \bar{v} = \frac{x_2 - x_1}{t_2 - t_1} \]

For velocity at a specific time \(t\), we use calculus:

\[ v(t) = \frac{dx}{dt} = \frac{d}{dt} x(t) = \frac{d^2x}{dt^2} = a \]

So, position: \(x\)
velocity: rate of change of position
acceleration: rate of change of velocity

The "(t)" part is left implicit sometimes.
\( x(t) \) contains all the information, but sometimes you know \( a \) or \( v \) and want to find \( x \).

For instance...

**Free-fall**

* Dropping demos, including feather & penny in vacuum chamber

\[
a(t) = a = -g \quad \text{on Earth}
\]

\[\uparrow \text{up} \quad 9.8 \text{ m/s}^2\]

* Careful!

If \( a(t) = \text{const.} \), what's \( v(t) \), \( x(t) \)? Calculus:

\[
v(t) = \int a(t) \, dt = \int a_0 \, dt = a_0 t + \text{const}
\]

\[\uparrow \text{"initial velocity"}
\]

since \( v(t=0) = v_0 \)

Then:

\[
x(t) = \int v(t) \, dt = \int (a_0 t + v_0) \, dt
\]

\[\uparrow \text{"initial position"}
\]

Generic result for \( a = \text{constant} \):

\( x(t) \) tells us everything about the path.
Example

\[ y(t) = y_0 + v_0 t + \frac{1}{2} g t^2 \]

\[ y(t) = y_0 - \frac{1}{2} g t^2 \]

\[ t_{\text{crash}} = \text{when } y(t) = 0, \text{ so:} \]

\[ 0 = y_0 - \frac{1}{2} g (t_{\text{crash}})^2 \]

\[ t_{\text{crash}} = \sqrt{\frac{2 y_0}{g}} \]

\[ t_{\text{crash}} = \sqrt{\frac{2 (10\text{ m})}{9.8\text{ m/s}^2}} \]

\[ t_{\text{crash}} = 1.43 \text{ s} \]

\[ v(t) = \frac{dy}{dt} = -gt \]

\[ v(t_{\text{crash}}) = (-9.8\text{ m/s}^2)(1.43\text{ s}) \]

\[ v(t_{\text{crash}}) = -14.0\text{ m/s} = -31.3\text{ mph} \]

2D motion

"static", "moving" are relative concepts. Can only say bull is moving relative to observer.

Also: laws of nature do not depend on motion (non-accelerated) system or observer.

So →
Can describe ball from observer's point of view:

- Ball falling with observer moving left at speed $v$ OR
- Stationary observer watching ball falling while also moving right with speed $v$

Also: rightward motion of ball didn't mess up downward motion.

**Demo:** car on track

### Trajectories

Take some freefall system:

- $y(t) = \frac{1}{2}at^2 + v_{0y}t + y_0$  \hspace{1cm} (9)  
  - $a$ = initial vel. in $y$ direction

If also some initial $x$ velocity $v_{0x}$:

- $x(t) = v_{0x}t + x_0$  

What is $y(t)$?  
(The trajectory)
Choose more convenient coordinate system:

\[
\begin{align*}
X &= V_{x0}t \\
Y &= -\frac{1}{2}gt^2 + V_{y0}t
\end{align*}
\]

\[\Rightarrow \quad \text{Solving:} \quad t = \frac{X}{V_{x0}} \quad \text{and} \quad Y = \left[ -\frac{1}{2} \frac{g}{V_{x0}^2} \right] X^2 + \left[ \frac{V_{y0}}{V_{x0}} \right] X \quad \text{a parabola!}
\]

\[\uparrow \quad \text{now implicitly} \quad "Y(X)" \]

Can ask questions about path:

What's the "range" \( x_R \)?

Need \( y = 0 \), so

\[\begin{align*}
\Delta &= \left[ -\frac{1}{2} \frac{g}{V_{x0}^2} \right] X^2 + \left[ \frac{V_{y0}}{V_{x0}} \right] X \\
\Rightarrow \quad &X = 0 \\
\text{or} \\
&X = \frac{2V_{x0}V_{y0}}{g} = x_R
\end{align*}\]

What angle \( \theta \) maximizes \( x_R \)?

\[V_x = V_0 \cos \theta \quad \text{and} \quad V_y = V_0 \sin \theta\]

\[\Rightarrow \quad x_R = \frac{2V_0^2 \cos \theta \sin \theta}{g}\]

\[= \frac{V_0^2}{g} \sin 2\theta \quad \Rightarrow \quad x_R \text{ max if } \sin 2\theta = 1 \]

\[\Rightarrow \theta = 45^\circ\]
Or, with calculus:

\[ \frac{dx_2}{d\theta} = 0 \Rightarrow \frac{2v_0^2}{g} \cos 2\theta = 0 \]

\[ \Rightarrow \cos 2\theta = 0 \]

\[ \Rightarrow \theta = 45^\circ \]

\[ \frac{d^2x_2}{d\theta^2} = \frac{-4v_0^2 \sin 2\theta}{g} \bigg|_{\theta=45^\circ} < 0 \Rightarrow \text{maximum, not minimum} \]

\[ \theta \]

A demo: monkey shoot

<table>
<thead>
<tr>
<th>no gravity</th>
<th>with gravity</th>
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Gravity adds \( \frac{1}{2}gt^2 \) to both \( y(t) \) functions, so will still collide