Prob 1 (20 points)  The strong interactions are blind to flavor. The $\Sigma$ triplet of baryons only differ in the number of $u$ and $d$ quarks so their mass difference should be due to two effects: the electromagnetic energy and the difference in mass between the $u$ and $d$ quarks.

(a) Calculate the EM energy for each of these baryons using the following simple model: assume that the three quarks are at average 1 fm from each other in the $\Sigma$. Express your answer in MeV.

(b) Calculate the gravitational energy for the three quarks in this model. For simplicity assume that each quark has about 1/3 the mass of the $\Sigma$ (use MeV again).

(c) Your answer to (a) is not very accurate since we really don’t know if 1 fm is the right distance. The relative binding energies maybe more reliable. Use them ad the three $\Sigma$ masses to solve for both the EM binding and the $d$-$u$ mass difference.

(d) The $K^0$-$K^\pm$ mass difference is $\sim 4$ MeV/$c^2$ and the $B^0$-$B^\pm$ is $0.1 \pm 0.8$ MeV/$c^2$. Does this suggest to you that the $K$-mesons are larger or smaller in size than the $B$ mesons?

Prob 2 (10 points)

The Higgs boson is a spin 0 particle which will decay primarily into $W$ and $Z$ pairs if its mass is sufficiently high. The coupling will be proportional to the Higgs mass divided by the $W$ mass, i.e. the vertex factor will be proportional to $-igM_H/M_W$ where $g$ is a dimensionless number. From dimensional analysis, how do you expect the Higgs decay width to vary with the unknown Higgs mass in this high mass region? (Hint: take the $|\mathcal{M}|^2$, i.e. the amplitude square)

Prob 3 (30 points) We define the “valence” quark distribution of the nucleon as by $u_c(x) \equiv u(x) - \bar{u}(x)$ and $d_c(x) \equiv d(x) - \bar{d}(x)$ and the “sea” contribution $u_s(x) = \bar{u}(x)$, $d_s(x) = \bar{d}(x)$, $s_s(x) = \bar{s}(x)$. In this problem we will take the $c$, $b$ and $t$ quarks so heavy that the probability of finding one in the proton is negligible (this is only an approximation). (a) what is $\int_0^1 u_v(x)dx$ and $\int_0^1 d_v(x)dx$ for the proton?

(b) Using the following proton quark distribution functions:
\[ x u_v(x) = 1.78 x^{0.5}(1 - x^{1.5})^{3.5} \]  
\[ x d_v(x) = 0.70 x^{0.4}(1 - x^{1.4})^{4.5} \]  
\[ x u_s(x) = x \bar{u}_s(x) = x d_s(x) = x \bar{d}_s(x) = 0.23(1 - x)^7 \]  
\[ x s(s) = x \bar{s}(x) = 0.1(1 - x)^7 \]  

What fraction of the proton momentum is carried by valence quarks, non-strange sea quarks and antiquarks, strange sea quarks and antiquarks and gluons?

(c) From DIS \( e^- p \rightarrow e^- X \) we study the \( F_2(x) = x \sum_i Q_i^2 f_i(x) \). We use the study of \( e^- (p + n) \rightarrow e^- X \) so that \( \sigma(e^- n) \sim \sigma(e^- d) - \sigma(e^- p) \).

Take

\[ F_2^p = x \left[ \frac{2}{3} (u(x) + \bar{u}(x)) + \left( \frac{1}{3} \right)^2 (d(x) + \bar{d}(x) + s(x) + s(\bar{x})) \right] \]

and

\[ F_2^n = x \left[ \frac{2}{3} (d(x) + \bar{d}(x)) + \left( \frac{1}{3} \right)^2 (u(x) + \bar{u}(x) + s(x) + s(\bar{x})) \right] \]

and calculate

\[ \frac{F_2^n}{F_2^p} \]

in the limit of \( x \rightarrow 1 \) and \( x \rightarrow 0 \).

**Prob 4 (40 points)** The expression for the Drell-Yan process \( pp \rightarrow e^+ e^- X \) differential cross section is

\[
\frac{d\sigma}{dM^2} = \frac{4\pi\alpha^2}{9M^2} \sum_q Q_q^2 \int_0^1 dx_q \int_0^1 dx_q f_q(x_q) f_{\bar{q}}(x_{\bar{q}}) \delta(M^2 - x_q x_{\bar{q}} s)
\]

, where \( M \) is the invariant mass of the \( e^+ e^- \) pair, \( Q_q \) is the quark charge in units of electron charge, and \( s \) is the square of \( pp \) center of mass energy. Use
the structure functions above to get an approximate expression for $d\sigma/dM^2$ at an LHC with $s = 14$ TeV and the one with $s = 7$ TeV. 

**Suggestion:**
The $x$'s involved here are quite small, so the valence quarks can be ignored. The $(1 - x)^7$ term in the sea quark distribution will be effectively unity for small $x$, so for example $u_s(x) \sim 0.23x$. Two warnings: (i) the lower limit of the integral can not be really 0. Since it only comes in a logarithm, it does not matter too much what you set it to, but set it to something reasonable. (ii) You will need the equation for transforming Dirac delta functions:

$$\delta(M^2 - x_q x_{\bar{q}} s) = \frac{1}{x_q} \delta(x_q - \frac{M^2}{x_{\bar{q}} s})$$

**Prob 5 (40 points)**
The origin $x = y = z = 0$ is at the nominal collision point in the geometrical center of the detector. The $z$ direction is the direction of the proton beam; the detector solid angle segmentation is designed to be invariant under boost along the $z$ direction. $\phi$ is the azimuthal angle about the axis.

(a) Show that a particle’s 4-momentum can be written as

$$p = (m_T \cosh Y, p_x, p_y, m_T \sinh Y)$$

where the transverse mass $m_T^2 = m^2 + p_x^2 + p_y^2$ and $Y$ is the boost parameter to the frame where the particle has 0 momentum along the $z$-axis. $Y$ is called the rapidity of the particle. Show that

$$Y = \tanh^{-1}(p_z/E) = \frac{1}{2} \ln \frac{E + p_z}{E - p_z}$$

(b) $\eta$ is the pseudorapidity, which is related to the polar angle $\theta$ by the relation:

$$\eta \equiv -\ln \tan \frac{\theta}{2}$$

The pseudorapidity $\eta$ is equivalent to the rapidity of a particle in the limit of $p >> m$, where $p$ is the momentum of the particle and $m$ is its mass. At $\theta = 90^\circ$ show that $\Delta \eta = \Delta \theta$. What is $\Delta \eta/\Delta \theta$ at $\eta = 3.0$?

Note that pseudorapidity depends only on the polar angle and is approximately Lorentz invariant under $z$-boosts for high $p_T$ particles. It is used to define angular segmentation. Being tranverse to the $z$-direction the $\phi$ angle is also invariant under $z$-boosts and is the orthogonal solid angle variable.
(c) Express $Y$ in terms of $\eta$ and $\alpha = m/\sqrt{p_T}$ where $p_T$ is the transverse momentum.

(d) Show that jets are approximately circular in $\eta$-$\phi$. Hint: consider a point in $\eta$-$\phi$ with momentum vector $\vec{P}$. Around it consider a set of points with small transverse momentum relative to $\vec{P}$ (representing the decay products produced symmetrically about the parton direction).
Prob 6 Razor Kinematics (50 points) The selectron is the hypothetical heavy superpartner of the electron. In principle we could produce a pair of oppositely-charged selectrons in a collision at the LHC. Consider the simple case where the two selectrons are produced at threshold, so they are at rest in the center-of-mass frame of the collision, and we can write their 4-momenta as:

\[ p_s^+ = (m_s, 0, 0, 0) \]
\[ p_s^- = (m_s, 0, 0, 0) \]

Each selectron decays to an electron (or positron) and a heavy neutralino dark matter particle \( \chi \). Use the approximation that the electron is massless, and denote the neutralino mass by \( m_\chi \). In the rest frame of the decaying selectron, the electron and the neutralino have 3-momenta of equal magnitude and opposite direction.

(a) Show that the magnitude of the 3-momentum of the electron (or positron), as measured in the center-of-mass frame, is given by
\[ |\tilde{p}_e| = m_\Delta/2 \]
where
\[ m_\Delta = \frac{m_s^2 - m_\chi^2}{m_s} \]

In an LHC collision the true center-of-mass frame is not known, but it is often a good approximation to say that it is related to the lab frame by a longitudinal boost, i.e. a boost along the beam axis, taken to be the \( z \) direction.

(b) Taking the electron/positron to be massless, show that we can always perform a longitudinal boost from the lab frame to the razor frame, defined as the frame where the \( z \)-components of the electron/positron momenta are equal and opposite, i.e. \( p_{e-}^z + p_{e+}^z = 0 \). Show that the corresponding boost parameter is given by
\[ \beta = \frac{p_{e-}^z + p_{e+}^z}{|\tilde{p}_{e-}| + |\tilde{p}_{e+}|} \]

Consider the razor kinematic variable \( M_R \). For the process we are considering, it is defined in the lab frame as:
\[ M_R = \sqrt{(|\tilde{p}_{e-}| + |\tilde{p}_{e+}|)^2 - (p_{e-}^z + p_{e+}^z)^2} \]
(c) Show that $M_R$ has the same form in any frame related to the lab frame by a longitudinal boost.

(d) Suppose that the razor frame and center-of-mass frame were the same frame. Then with the approximations that we made above, compute the value of $M_R$. Make an argument that more generally the distribution of values of $M_R$ for many collisions should peak at this same value.

The neutralinos from the selectron decays are not observed. However we can measure the imbalance in the total transverse momentum of the electron-positron pair, caused by the fact that they are recoiling from two neutralinos. This is a 2-vector in the $x$-$y$ plane transverse to the beam axis, defined as

$$\vec{p}_{T\text{miss}} = -(\vec{p}_{T e^-} + \vec{p}_{T e^+}) = - (p_{T e^-}^x + p_{T e^+}^x, p_{T e^-}^y + p_{T e^+}^y)$$

Of course transverse quantities are invariant under longitudinal boosts. Now we can define the transverse razor variable $M_R^{T}$ by

$$(M_R^{T})^2 = \frac{1}{2} [ |\vec{p}_{T\text{miss}}| (|\vec{p}_{T e^-}| + |\vec{p}_{T e^+}|) - \vec{p}_{T\text{miss}} \cdot (\vec{p}_{T e^-} + \vec{p}_{T e^+}) ]$$

(e) Using the same approximations that we did above, show that the largest possible value for $M_R^{T}$ is $m_\Delta$ (i.e. this second razor variable has a kinematic endpoint at $m_\Delta$).
**Prob 7** (Collider Cost, 20 points) In a simplified model, the cost, C, of an $e^+e^-$ storage ring collider is given by

$$C = \alpha + \beta R + \gamma \frac{E^4}{R}$$  

where R is the radius of the ring, E is the energy, and $\alpha$, $\beta$, $\gamma$ are some constants that depend on the technology. The first term represents the fixed costs, such as the control room; the second term represents the items that depend on the length of the ring such as the magnets and vacuum system; and the third term represents the cost of the RF system, assuming that its cost is proportional to the amount of beam power lost per turn by synchrotron radiation. Show that for any fixed technology, both the cost and radius of an $e^+e^-$ storage ring scale as the energy squared. (Burton Richter, then director of SLAC, used this argument in a paper he wrote in 1976 to argue that at some very high energy, a linear $e^+e^-$ collider, in which the costs scale linearly with energy, must be more cost effective than a circular accelerator.)

**Prob 8: EXTRA CREDIT** (error propagation and accuracy, 30 points) Magnetized iron is often used to measure the momentum of muons. It can be easily magnetized to about 1.8 Tesla, where the ferromagnetic properties of the iron saturate. Consider a relativistic muon ($\beta \sim 1$) of momentum $p$ normally incident on a block of iron $d$ meters thick that has been magnetized to 1.8 T perpendicular to the muons direction. (Assume that the radius of magnetic curvature is large compared to the thickness of the iron block.)

(a) What is the angle of deflection due to the magnetic field? (b) What is the rms angle of deflection due to multiple Coulomb scattering? (c) From parts (a) and (b), what is the maximum rms precision with which the muon momentum can be determined by measuring the angle of deflection? (d) Suppose you divide the block of iron into two blocks, each of thickness $d/2$, and measure the angle of deflection in each of the two blocks. Do you gain in overall precision?