MAJOR OPEN QUESTIONS IN PHYSICS

Problem Set 6
Due Tuesday, February 24, 2004

1. As explained in class, the renormalizability of field theories can be characterized by the mass dimension of the coupling constant:

i. **Super-Renormalizable theory:** Coupling constant has positive mass dimension.

ii. **Renormalizable theory:** Coupling constant is dimensionless.

iii. **Non-Renormalizable theory:** Coupling constant has negative mass dimension.

For what spacetime dimensions $d$ is gravity in category i / ii / iii? Treat $d$ as a (non-negative) real number, and give closed or open intervals.

2. Write down the amplitude (no need to solve it or do any integrals, just write the formula down) for each of the following Feynman diagrams (call the incoming and outgoing momenta $p_1, p_2, ...$):

a) \[ \text{(Möller scattering)} \]

b) \[ \text{(Compton scattering)} \]

c) \[ \text{(Bremsstrahlung)} \]

d) \[ \text{(pair annihilation)} \]

e) \[ \text{(Vortex correction)} \]

f) \[ \text{(Vacuum polarization)} \]

Which of the above are ultraviolet divergent?
3. **Variational problem for strings.** Consider a string stretched from \( x = 0 \) to \( x = a \), with a tension \( T_0 \) and a position-dependent mass density \( \mu(x) \). The string is fixed at the endpoints and can vibrate in the \( y \)-direction. The equation
\[
\frac{d^2 y}{dx^2} + \frac{\mu(x)}{T_0} \omega^2 y(x) = 0
\]
determines the oscillation frequencies \( \omega_i \) and associated profiles \( \psi_i(x) \) for this string.

a. Set up a variational procedure that gives an upper bound on the lowest frequency of oscillation \( \omega_0 \). (This can be done roughly as in quantum mechanics, where the ground state energy \( E_0 \) of a system with Hamiltonian \( H \) satisfies \( E_0 \leq \langle \psi | H | \psi \rangle / \langle \psi | \psi \rangle \)). As a useful first step, consider the inner product
\[
\langle \psi_i | \psi_j \rangle \equiv \int_0^a \mu(x) \psi_i(x) \psi_j(x) \, dx
\]
and show that it vanishes when \( \omega_i \neq \omega_j \). Explain why your variational procedure works.

b. Consider the case \( \mu(x) = \mu_0 x \). Use your variational principle to find a simple bound on the lowest oscillation frequency. Compare with the answer \( \omega_0^2 \approx (18.956) \frac{T_0}{\mu_0 a^2} \) obtained by a direct numerical solution of the eigenvalue problem.