These notes collect and remind you of several definitions, connected with the notion of a Hilbert space.

**Def:** A (nonempty) set $V$ is called a **(complex) linear (or vector) space**, and its elements are called **vectors** if:

(a) An operation of “addition” is defined for every pair of elements $x \in V$ and $y \in V$, such that the “sum” $x + y \in V$.

(b) There exists a “zero vector,” $0 \in V$ such that $x + 0 = x$ for all $x \in V$.

(c) An operation of “multiplication” by a complex number (“scalar”) is defined so that, if $c$ is any complex number, and $x$ any vector in $V$, then the “product” $cx \in V$.

(d) The following properties (of ordinary vector algebra) are satisfied: ($x, y,$ and $z$ are any elements of $V$, and $c, c_1,$ and $c_2$ are any complex numbers)

\[
\begin{align*}
    x + y &= y + x & \text{1) commutativity} \\
    (x + y) + z &= x + (y + z) & \text{2) associativity} \\
    c_1(c_2x) &= (c_1c_2)x \\
    x + (-x) &= 0 & \text{3) inverse} \\
    1x &= x & \text{4) multiplication by scalar 1} \\
    (c_1 + c_2)x &= c_1x + c_2x & \text{5) distributivity} \\
    c(x + y) &= cx + cy.
\end{align*}
\]

If we restrict to real scalars, we have a “real vector space.” Note that the 0 element is unique by virtue of the fact that the vectors under the operation of addition form an abelian group.

**Def:** $m$ vectors $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ are **linearly dependent** if there exist $m$ constants, not all zero, such that

\[
\sum_{i=1}^{m} c_i x^{(i)} = 0
\]

Otherwise, the vectors are **linearly independent**.
**Def:** A linear space $V$ is *n-dimensional* if it contains a set of $n$ linearly independent vectors, but no set of more than $n$ linearly independent vectors.

**Def:** A set of linearly independent vectors, $e_1, e_2, e_3, \cdots$ in a vector space $V$ forms a **basis** for $V$ if, for any vector $x \in V$, we can find scalars $c_1(x), c_2(x), \cdots$ such that:

$$x = c_1(x)e_1 + c_2(x)e_2 + \cdots$$

**Def:** A **relation** is a set of ordered pairs.

**Def:** A **function** is a relation such that no two distinct members have the same first coordinate. (To a mathematician, the following terms are really synonymous: function, map, operator, transformation, correspondence).

**Def:** A linear space $V$ is called **pre-Hilbert**, or **Euclidean**, if a function is defined which assigns to every pair of vectors $x, y \in V$ a complex number $\langle x|y \rangle$, called the **scalar product** (or **inner product**) of $x$ and $y$, which satisfies the following properties:

1. $\langle x|x \rangle \geq 0$; $\langle x|x \rangle = 0$ iff $x = 0$
2. $\langle x|y \rangle = \overline{\langle y|x \rangle}$
3. $\langle cx|y \rangle = c\langle x|y \rangle$ (c is any complex number)
4. $\langle x|y_1 + y_2 \rangle = \langle x|y_1 \rangle + \langle x|y_2 \rangle$

**Def:** A non-empty set $M$ is called a **metric space** if to every pair of elements $x, y \in M$ there is assigned a real number $d(x, y)$ called the **distance** between $x$ and $y$, such that:

1. $0 \leq d(x, y)$ (and $< \infty$)
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)
Note that a metric space need not be a linear space. However, if we have a pre-Hilbert space, we may define a suitable distance according to:

\[ d(x, y) = \sqrt{\langle (x - y)(x - y) \rangle} \]

We will typically deal only with metric spaces which are also pre-Hilbert spaces. We will also use the notation

\[ |x - y| = d(x, y) \]

for the distance function. We will furthermore define the “length,” or “norm,” of a vector by its distance from the zero vector: \( |x| = |x - 0| = d(x, 0) = \sqrt{\langle x|x \rangle} \). Also, if \( \langle x|y \rangle = 0 \), we say that \( x \) is “orthogonal” to \( y \).

**Def:** A sequence of elements in a metric space, \( x_1, x_2, \ldots \) is said to **converge** to an element \( x \) if, given \( \epsilon > 0 \), there exists a number \( N \) such that:
\[ |x - x_n| < \epsilon \quad \text{whenever} \quad n > N \]
In this case, we write \( x = \lim_{n \to \infty} x_n \).

**Def:** A sequence of elements in a metric space is called a **Cauchy sequence** if, given \( \epsilon > 0 \), there exists \( N \) such that:
\[ |x_n - x_m| < \epsilon \quad \text{for all} \quad n, m > N \]
(“Cauchy Convergence Criterion”)

**Theorem:** Every convergent sequence of elements in a metric space is a Cauchy sequence.

**Proof:** Suppose \( x_1, x_2, \ldots \) is a convergent sequence of elements such that \( \lim_{n \to \infty} x_n = x \). Then, given any \( \epsilon > 0 \), we may find a number \( N \) such that:
\[ |x - x_m| < \frac{1}{2} \epsilon \quad \text{for all} \quad n > N \]
Consider
\[ |x_n - x_m| = |x_n - x + x - x_m| \]
\[ \leq |x_n - x| + |x - x_m| \quad \text{(triangle inequality)} \]
\[ < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon \quad \text{for all} \quad n, m > N \]
\[ < \epsilon \]

QED
Def: A metric space \((V, d)\) is said to be complete if every Cauchy sequence of points in \(V\) converges to a point in \(V\).

Theorem: (and Definition) If \((V, d)\) is an incomplete metric space, there exists a complete metric space \((V^*, d^*)\) called the completion of \(V\) which corresponds to an isometric (i.e., distance preserving) mapping of \(V\) into \(V^*\) such that the closure (i.e., the intersection of all closed sets containing \(V\)) of the image of \(V\) coincides with \(V^*\). (For an instructive, but mildly lengthy proof, see: Fano, Mathematical Methods of Quantum Mechanics.)

Note that this theorem tells us that every pre-Hilbert space admits a completion, which we call a Hilbert space. If we wish to continue this discussion, for example to construct a suitable Hilbert space of functions for quantum mechanical problems, we will properly need to consider measure theory, etc. However, let us turn our attention now to other matters.

Def: Suppose that we have two vector spaces, \(V\) and \(V'\), and that there exists a correspondence which assigns to every vector \(x \in D_A \subseteq V\), a vector \(x' \in V'\). We say that this correspondence defines an operator \(A\) from \(V\) into \(V'\) with domain \(D_A\), and write \(x' = Ax\).

The subset \(R_A\) of \(V'\) defined by

\[
R_A = \{x' \mid x' = Ax \text{ for some } x \in D_A\}
\]

is called the range of \(A\).

If \(V' = V\), then we say that \(A\) is defined in \(V\).

If \(D_A = V\), then we say that \(A\) is defined on \(V\).

If \(V'\) is the vector space formed by the complex numbers (with ordinary operations of addition and multiplication by a complex number), we often use the term functional instead of operator, and write \(x' = f(x)\) rather than \(x' = Ax\).

Def: Two operators \(A\) and \(B\) from \(V\) into \(V'\) are said to be equal if \(D_A = D_B\) and
Ax = Bx \quad \text{for all } x \in D_A

If, on the other hand, \( D_B \) is a proper subset of \( D_A \), and \( Ax = Bx \) for all \( x \in D_B \), we call \( A \) an **extension** of \( B \), and \( B \) a **restriction** of \( A \). We may denote this by writing \( B \subset A \).

**Def:** An operator \( L \) is said to be **linear** if its domain \( D_L \) is a subspace of \( V \) (i.e. \( D_L \) is a vector space) and

\[
L(x + y) = Lx + Ly \quad \text{for all } x, y \in D_L
\]
\[
L(cx) = cLx \quad \text{for all } x \in D_L \text{ and for all scalars } c
\]

Henceforth, we will typically mean “linear operator” whenever we say “operator”.