Solution: chapter 4, problem 2:

Let's start by more formally repeating what we are to show. We consider a sample $\mathbf{x} = \{x_1, \ldots, x_N\}$ drawn i.i.d. from (unknown) pdf $f(x)$. The sample mean is

$$m = \frac{1}{N}\sum_{n=1}^{N} x_n. \tag{4.1}$$

The sample mean provides an estimate for the mean $E(X)$. Let b be the bias of the sample mean as an estimate of the mean:

$$b \equiv E\left[m - E(X)\right]. \tag{4.2}$$

Now suppose we wish to use the jackknife to estimate this bias using our dataset. Denote the jackknife bias estimate by $\widehat{b}_N$, following the notation in section 4.3 of Narsky and Porter (2014) (referred to hereafter as NP). We are supposed to show that $\widehat{b}_N = 0$.

NP Eq. 4.19 tells us that the jackknife bias estimate is

$$\widehat{b}_N = (N-1)(m' - m), \tag{4.3}$$

where

$$m' \equiv \frac{1}{N}\sum_{i=1}^{N} m_{-i}, \tag{4.4}$$

and

$$m_{-i} \equiv \frac{1}{N-1}\sum_{n\neq i} x_n. \tag{4.5}$$

Expanding Eq. 4.3,

$$\begin{aligned}
\widehat{b}_N &= (N-1)\left[\frac{1}{N}\sum_{i=1}^{N} m_{-i} - m\right] \\
&= (N-1)\left[\frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{n\neq i} x_n - m\right] \\
&= (N-1)\left[\frac{1}{N(N-1)}\sum_{i=1}^{N}(Nm - x_i) - m\right] \\
&= (N-1)\left[\frac{1}{N-1}(Nm - m) - m\right] \\
&= 0.
\end{aligned} \tag{4.6}$$

Thus, we have demonstrated that the jackknife estimate for the bias of the sample mean, as an estimator for the mean, is zero, which is what we were asked to show.

In the development of the jackknife estimates in NP section 4.3 two assumptions are made:

1) Our estimator (in this case m for $E(X)$) is consistent.
2) The approach to $E(X)$ is approached at leading order as $1/N$, or rather

$$b_{N-1} \propto \frac{N}{N-1} b_N, \text{ for large enough } N. \tag{4.7}$$

We should check the validity of these assumptions. For the present case, the bias is in fact zero: $b_N = E\left[m - E(X)\right] = 0$, and our estimate of the mean is therefore also consistent. The second condition is also trivially satisfied.

Bibliography

Narsky, I. and Porter, F.C. (2014) *Statistical Analysis Techniques in Particle Physics*, Wiley-VCH.